

Quantum Gravity without General Relativity

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Abstract

The quantum field theory of gravitation is constructed in terms of Lagrangian density of Dirac fields which couple to the electromagnetic field A_μ as well as the gravitational field \mathcal{G} . The gravity appears in the mass term as $m(1 + g\mathcal{G})\bar{\psi}\psi$ with the coupling constant of g . In addition to the gravitational force between fermions, the electromagnetic field A_μ interacts with the gravity as the fourth order effects and its strength amounts to α times the gravitational force. Therefore, the interaction of photon with gravity is not originated from Einstein's general relativity which is entirely dependent on the unphysical assumption of the principle of equivalence. Further, we present a renormalization scheme for the gravity and show that the graviton stays massless.

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1 Introduction – Problems of General Relativity

The motion of the earth is governed by the gravitational force between the earth and the sun, and the Newton equation is written as

$$m\ddot{\mathbf{r}} = -G_0mM\frac{\mathbf{r}}{r^3} \quad (1.1)$$

where G_0 , m and M denote the gravitational constant, the mass of the earth and the mass of the sun, respectively. This is the classical mechanics which works quite well. Further, Einstein generalizes the Newton equation to the relativistic equation of motion which can be valid even for the curved space [1, 2]. However, this was achieved before the discovery of quantum mechanics, and therefore it is natural that the general relativity cannot be quantized properly. Indeed, the quantization of the general relativity has intrinsic problems which are related to the invariance of the general coordinate transformation. On the other hand, the first quantization is only possible for the Cartesian coordinates [3]. This indicates that any attempt to quantize the general relativity is not a proper starting point, but rather one should try to make a field theory simply to include the gravity. This is closely connected to the understanding of the first quantization ($[x_i, p_j] = i\hbar\delta_{ij}$), and since this quantization procedure is not a fundamental principle, we should try to make a field theory which includes the gravitational interaction [3, 4].

Before constructing a theory that can describe the field equation under the gravity, we discuss the fundamental problems in the theory of general relativity. Basically, there are two serious problems in the general relativity, the lack of field equation under the gravity and the assumption of the principle of equivalence.

1.1 Field Equation of Gravity

When one wishes to write the Dirac equation for a particle under the gravitational interaction, then one faces to the difficulty. Since the Dirac equation for a hydrogen-like atom can be written as

$$\left(-i\nabla \cdot \boldsymbol{\alpha} + m\beta - \frac{Ze^2}{r}\right) \Psi = E\Psi \quad (1.2a)$$

one may write the Dirac equation for the gravitational potential $V(r) = -\frac{G_0mM}{r}$ as

$$\left(-i\nabla \cdot \boldsymbol{\alpha} + m\beta - \frac{G_0mM}{r}\right) \Psi = E\Psi. \quad (1.2b)$$

But there is no foundation for this equation. At least, one cannot write the Lagrangian density which can describe the Dirac equation for the gravitational interaction. This is clear since one does not know whether the interaction can be put into the zero-th component of a vector type or a simple scalar type in the Dirac equation. That is, it may be of the following type

$$\left[-i\nabla \cdot \boldsymbol{\alpha} + \left(m - \frac{G_0mM}{r}\right)\beta\right] \Psi = E\Psi. \quad (1.2c)$$

This is a well known problem, but it is rarely discussed, and people seem to be reluctant to treating this problem up on the table.

1.2 Principle of Equivalence

The theory of general relativity is entirely based on the principle of equivalence. Namely, Einstein started from the assumption that physics of the two systems (a system under the uniform external gravity and a system that moves with a constant acceleration) must be the same. This looks plausible from the experience on the earth. However, one can easily convince oneself that the system that moves with a constant acceleration cannot be defined properly since there is no such an isolated system in a physical world. The basic problem is that the assumption of the principle of equivalence is concerned with the two systems which specify space and time, not just the numbers in connection with the acceleration of a particle. Note that the acceleration of a particle is indeed connected to the gravitational acceleration, $\ddot{z} = -g$, but this is, of course, just the Newton equation. Therefore, the principle of equivalence inevitably leads Einstein to the space deformation. It is clear that physics must be the same between two inertia systems, and any assumption which contradicts this basic principle cannot be justified at all.

Besides, this problem can be viewed differently in terms of Lagrangian. For the system under the uniform external gravity, one can write the corresponding Lagrangian. On the other hand, there is no way to construct any Lagrangian for the system that moves with a constant acceleration. One can define a Lagrangian for a particle that moves with a constant acceleration, but one cannot write the system (or space and time) that moves with a constant acceleration. Therefore, it is very hard to accept the assumption of the principle of equivalence even with the most modest physical intuition.

1.3 General Relativity

Einstein generalized the classical mechanics to the relativistic equation of motion where he started from the principle of equivalence. Therefore, he had to introduce the new concept that space may not be uniform, and the general relativity is the equation for the metric tensor $g_{\mu\nu}$. However, this picture is still based on the particle mechanics which is governed by the equations for coordinates of a point particle. This is, of course, natural for Einstein since quantum mechanics was not discovered at that time. Since quantum mechanics is a field theory, though with the non-relativistic kinematics, it is essentially different from Newton's classical mechanics but is rather similar to the Maxwell equations. Newton equation can certainly describe the dynamics of particles for the certain region of kinematics such as the motion of the earth around the sun. However, it is a useless theory for the description of electron motion in atoms. One should give up the idea of particle picture and should accept the concept of field theory. At the time of invention of the general relativity, Einstein knew quite well the Maxwell equations which are indeed field theory equations. However, the Maxwell equations are not realized as the basis equations for quantum mechanics [3].

2 Lagrangian Density for Gravity

It is by now clear that one should start from constructing the quantum mechanics of the gravitation. In other words, one should find the Dirac equation for electron when it moves in the gravitational potential. In this paper, we present a model Lagrangian density which can describe electrons interacting with the electromagnetic field A_μ as well as the gravitational field \mathcal{G} .

2.1 Lagrangian Density for QED

We first write the well established Lagrangian density for electrons interacting with the electromagnetic field A_μ

$$\mathcal{L}_{el} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This Lagrangian density of QED is best studied and is most reliable in many respects. In particular, the renormalization scheme of QED is theoretically well understood and is experimentally well examined, and there is no problem at all in the perturbative treatment of QED. All the physical observables can be described in terms of the free Fock space terminology after the renormalization, and therefore one can compare any prediction of the physical quantities with experiment. However, it should be noted that QED is the only field theory model in four dimensions which works perfectly well without any conceptual difficulties.

2.2 Lagrangian Density for QED plus Gravity

Now, we propose to write the Lagrangian density for electrons interacting with the electromagnetic field as well as the gravitational field \mathcal{G}

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - m(1 + g\mathcal{G})\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\partial^\mu\mathcal{G} \quad (2.2)$$

where the gravitational field \mathcal{G} is assumed to be a massless scalar field. It is easy to prove that the new Lagrangian density is invariant under the local gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu\chi, \quad \psi \rightarrow e^{-ie\chi}\psi. \quad (2.3)$$

This is, of course, quite important since the introduction of the gravitational field does not change the most important local symmetry.

2.3 Dirac Equation with Gravitational Interactions

Now, one can easily obtain the Dirac equation for electrons from the new Lagrangian density

$$i\gamma^\mu \partial_\mu \psi - e\gamma^\mu A_\mu \psi - m(1 + g\mathcal{G})\psi = 0. \quad (2.4)$$

Also, one can write the equation of motion of gravitational field

$$\partial_\mu \partial^\mu \mathcal{G} = -mg\bar{\psi}\psi. \quad (2.5)$$

The symmetry property of the new Lagrangian density can be easily examined, and one can confirm that it has a right symmetry property under the time reversal transformation, parity transformation and the charge conjugation [3].

2.4 Total Hamiltonian for QED plus Gravity

The Hamiltonian can be constructed from the Lagrangian density in eq.(2.2)

$$\begin{aligned} H = \int \{ \bar{\psi} (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m(1 + g\mathcal{G})) \psi - e\mathbf{j} \cdot \mathbf{A} \} d^3r + \frac{e^2}{8\pi} \int \frac{j_0(\mathbf{r}')j_0(\mathbf{r})d^3rd^3r'}{|\mathbf{r}' - \mathbf{r}|} \\ + \frac{1}{2} \int \left(\dot{\mathbf{A}}^2 + (\boldsymbol{\nabla} \times \mathbf{A})^2 \right) d^3r + \frac{1}{2} \int \left(\dot{\mathcal{G}}^2 + (\boldsymbol{\nabla}\mathcal{G})^2 \right) d^3r \end{aligned} \quad (2.6)$$

where j_μ is defined as $j_\mu = \bar{\psi}\gamma_\mu\psi$. In this expression of the Hamiltonian, the gravitational energy is still written without making use of the equation of motion. In the next section, we will treat the gravitational energy and rewrite it into an expression which should enable us to easily understand the structure of gravitational force between fermions.

3 Static-dominance Ansatz for Gravity

In eq.(2.2), the gravitational field \mathcal{G} is introduced as a *real scalar* field, and therefore it cannot be a physical observable as a classical field [5]. In this case, since the real part of the right hand side in eq.(2.5) should be mostly time independent, it may be reasonable to assume that the gravitational field \mathcal{G} can be written as the sum of the static and time-dependent terms and that the static part should carry the information of diagonal term in the external source term. Thus, the gravitational field \mathcal{G} is assumed to be written as

$$\mathcal{G} = \mathcal{G}_0(\mathbf{r}) + \bar{\mathcal{G}}(x) \quad (3.1)$$

where $\mathcal{G}_0(\mathbf{r})$ does not depend on time. This ansatz is only a sufficient condition, and its validity cannot be verified mathematically, but it can be examined experimentally.

The equations of motion for $\mathcal{G}_0(\mathbf{r})$ and $\bar{\mathcal{G}}(x)$ become

$$\nabla^2 \mathcal{G}_0 = mg\rho_g \quad (3.2)$$

$$\partial_\mu \partial^\mu \bar{\mathcal{G}}(x) = -mg\{(\bar{\psi}\psi)_{[\text{non-diagonal}]} + (\bar{\psi}\psi)_{[\text{diagonal rest}]}\} \quad (3.3)$$

where ρ_g is defined as

$$\rho_g \equiv (\bar{\psi}\psi)_{[\text{diagonal}]} \quad (3.4)$$

where $(\bar{\psi}\psi)_{[\text{diagonal}]}$ denotes the diagonal part of the $\bar{\psi}\psi$, that is, the terms proportional to $[a_{\mathbf{k}}^{\dagger(s)} a_{\mathbf{k}'}^{(s)} - b_{\mathbf{k}}^{\dagger(s)} b_{\mathbf{k}'}^{(s)}]$ of the fermion operators which will be defined in eq.(4.2). Further, $(\bar{\psi}\psi)_{[\text{non-diagonal}]}$ term is a non-diagonal part which is connected to the creation and annihilation of fermion pairs, that is, $[a_{\mathbf{k}}^{\dagger(s)} b_{-\mathbf{k}'}^{\dagger(s)} + b_{-\mathbf{k}'}^{(s)} a_{\mathbf{k}}^{(s)}]$ of the fermion operators. In addition, the term $(\bar{\psi}\psi)_{[\text{diagonal rest}]}$ denotes time dependent parts of the diagonal term in the fermion density, and this may also have some effects when the gravity is quantized. In this case, we can solve eq.(3.2) exactly and find a solution

$$\mathcal{G}_0(\mathbf{r}) = -\frac{mg}{4\pi} \int \frac{\rho_g(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3 r' \quad (3.5)$$

which is a special solution that satisfies eq.(2.5), but not the general solution. Clearly as long as the solution can satisfy the equation of motion of eq.(2.5), it is physically sufficient. The solution of eq.(3.5) is quite important for the gravitational interaction since this is practically a dominant gravitational force in nature.

Here, we assume that the diagonal term of $(\bar{\psi}\psi)_{[\text{diagonal}]}$ is mostly time independent, and in this case, the static gravitational energy which we call H_G^S can be written as

$$\begin{aligned} H_G^S &= mg \int \rho_g \mathcal{G}_0 d^3 r + \frac{1}{2} \int (\nabla \mathcal{G}_0)^2 d^3 r \\ &= -\frac{m^2 G_0}{2} \int \frac{\rho_g(\mathbf{r}') \rho_g(\mathbf{r})}{|\mathbf{r}' - \mathbf{r}|} d^3 r d^3 r' \end{aligned} \quad (3.6a)$$

where the gravitational constant G_0 is related to the coupling constant g as

$$G_0 = \frac{g^2}{4\pi}. \quad (3.7)$$

This static gravitational energy can be written in the momentum representation as

$$H_G^S = -\frac{m^2 G_0}{4\pi^2} \sum_{\mathbf{p}, \mathbf{p}'} \int \frac{\bar{u}(\mathbf{p} + \mathbf{q}) u(\mathbf{p}) \bar{u}(\mathbf{p}' - \mathbf{q}) u(\mathbf{p}')}{q^2} d^3 q. \quad (3.6b)$$

Eq.(3.6) is just the gravitational interaction energy for the matter fields, and one sees that the gravitational interaction between electrons is always attractive. This is clear since the gravitational field is assumed to be a massless scalar. It may also be important to note that the H_G^S of eq.(3.6) is obtained without making use of the perturbation theory, and it is indeed exact, apart from the static ansatz of the field $\mathcal{G}_0(\mathbf{r})$.

4 Quantization of Gravitational Field

In quantum field theory, we should quantize fields. For fermion fields, we should quantize the Dirac field by the anti-commutation relations of fermion operators. This is required from the experiment in terms of the Pauli principle, that is, a fermion can occupy only one quantum state. In order to accommodate this experimental fact, we should always quantize the fermion fields with the anti-commutation relations. On the other hand, for gauge fields, we must quantize the vector field in terms of the commutation relation which is also required from the experimental observation that one photon is emitted by the transition between $2p$ -state and $1s$ -state in hydrogen atoms. That is, a photon is created from the vacuum of the electromagnetic field, and therefore the field quantization is an absolutely necessary procedure. However, it is not very clear whether the gravitational field \mathcal{G} should be quantized according to the bosonic commutation relation or not. In fact, there must be two choices concerning the quantization of the gravitational field \mathcal{G} .

4.1 No Quantization of Gravitational Field $\bar{\mathcal{G}}$

As the first choice, we may take a standpoint that the gravitational field \mathcal{G} should not be quantized since there is no requirement from experiments. In this sense, there is no definite reason that we have to quantize the scalar field and therefore the gravitational field \mathcal{G} should remain to be a classical field. In this case, we do not have to worry about the renormalization of the graviton propagator, and we obtain the gravitational interaction between fermions as we saw it in eq.(3.6) which is always attractive, and this is consistent with the experimental requirement.

4.2 Quantization Procedure

Now, we take the second choice and should quantize the gravitational field $\bar{\mathcal{G}}$. This can be done just in the same way as usual scalar fields

$$\bar{\mathcal{G}}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}} \left[d_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}} + d_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{r}} \right] \quad (4.1)$$

where $\omega_k = |\mathbf{k}|$. The annihilation and creation operators d_k and d_k^\dagger are assumed to satisfy the following commutation relations

$$[d_{\mathbf{k}}, d_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'} \quad (4.2)$$

and all other commutation relations should vanish. Since the graviton can couple to the time dependent external field which is connected to the creation or annihilation of the fermion pairs, the graviton propagator should be affected from the vacuum polarization of fermions. Therefore, we should carry out the renormalization procedure of the graviton propagator such that it can stay massless. We will discuss the renormalization procedure in the later chapter.

4.3 Graviton

Once the gravitational field \mathcal{G} is quantized, then the graviton should appear. From eq.(4.1), one can see that the graviton can indeed propagate as a free massless particle after it is quantized, and this situation is just the same as the gauge field case in QED, namely, photon after the quantization becomes a physical observable. However, it should be noted that the gauge field has a special feature in the sense that the classical gauge field (\mathbf{A}) is gauge dependent and therefore it is not a physical observable. After the gauge fixing, the gauge field can be quantized since one can uniquely determine the gauge field from the equation of motion, and therefore its quantization is possible.

On the other hand, the gravitational field is assumed to be a real scalar field, and therefore it cannot be a physical observable as a classical field [5]. Only after the quantization, it becomes a physical observable as a graviton, and this can be seen from eq.(4.1) since the creation of the graviton should be made through the second term of eq.(4.1). In this case, the graviton field is a complex field which is an eigenstate of the momentum and thus it is a free graviton state, which can propagate as a free particle.

5 Interaction of Photon with Gravity

From the Lagrangian density of eq.(2.2), one sees that photon should interact with the gravity in the fourth order Feynman diagrams as shown in Fig. 1. The interaction Hamiltonian H_I can be written as

$$H_I = \int (mg\mathcal{G}\bar{\psi}\psi - e\bar{\psi}\boldsymbol{\gamma}\psi \cdot \mathbf{A}) d^3r \quad (5.1)$$

where the fermion field ψ is quantized in the normal way

$$\psi(\mathbf{r}, t) = \sum_{\mathbf{p}, s} \frac{1}{\sqrt{L^3}} (a_{\mathbf{p}}^{(s)} u_{\mathbf{p}}^{(s)} e^{i\mathbf{p} \cdot \mathbf{r} - iE_{\mathbf{p}} t} + b_{\mathbf{p}}^{\dagger(s)} v_{\mathbf{p}}^{(s)} e^{-i\mathbf{p} \cdot \mathbf{r} + iE_{\mathbf{p}} t}) \quad (5.2)$$

where $u_{\mathbf{p}}^{(s)}$ and $v_{\mathbf{p}}^{(s)}$ denote the spinor part of the plane wave solutions of the free Dirac equation. $a_{\mathbf{p}}^{(s)}$ and $b_{\mathbf{p}}^{(s)}$ are annihilation operators for particle and anti-particle states, and they should satisfy the following anti-commutation relations,

$$\{a_{\mathbf{p}}^{(s)}, a_{\mathbf{p}'}^{\dagger(s')}\} = \delta_{s,s'} \delta_{\mathbf{p},\mathbf{p}'}, \quad \{b_{\mathbf{p}}^{(s)}, b_{\mathbf{p}'}^{\dagger(s')}\} = \delta_{s,s'} \delta_{\mathbf{p},\mathbf{p}'} \quad (5.3)$$

and all other anticommutation relations should vanish. The gauge field \mathbf{A} can be quantized in the same way

$$\mathbf{A}(x) = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \boldsymbol{\epsilon}^{\lambda}(\mathbf{k}) \left[c_{\mathbf{k},\lambda} e^{-ikx} + c_{\mathbf{k},\lambda}^{\dagger} e^{ikx} \right] \quad (5.4)$$

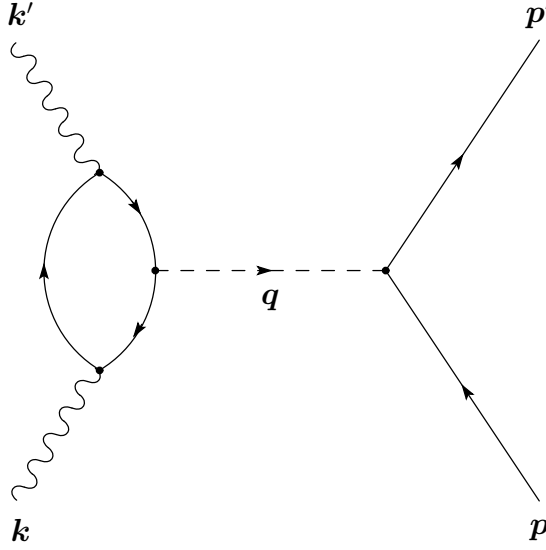


Fig. 1: The fourth order Feynman diagram

where $\omega_{\mathbf{k}} = |\mathbf{k}|$. The polarization vector $\epsilon^\lambda(\mathbf{k})$ should satisfy the following relations

$$\epsilon^\lambda(\mathbf{k}) \cdot \mathbf{k} = 0, \quad \epsilon^\lambda(\mathbf{k}) \cdot \epsilon^{\lambda'}(\mathbf{k}) = \delta_{\lambda, \lambda'}. \quad (5.5)$$

The annihilation and creation operators $c_{\mathbf{k}, \lambda}$, $c_{\mathbf{k}, \lambda}^\dagger$ should satisfy the following commutation relations

$$[c_{\mathbf{k}, \lambda}, c_{\mathbf{k}', \lambda'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\lambda, \lambda'} \quad (5.6)$$

and all other commutation relations should vanish.

The calculation of the S-matrix can be carried out in a straightforward way [6, 7, 8], and we can write

$$S = (ie)^2 \epsilon_\mu^\lambda(k) \epsilon_\nu^{\lambda'}(k') \left(\frac{mm'g^2}{q^2} \right) \bar{u}(p') u(p) \times \int \frac{d^4 a}{(2\pi)^4} \text{Tr} \left[\gamma_\mu \frac{i}{\not{a} - m + i\epsilon} \gamma_\nu \frac{i}{\not{b} - m + i\epsilon} \frac{i}{\not{c} - m + i\epsilon} \right] \quad (5.7)$$

where k and k' denote the four momenta of the initial and final photons while p and p' denote the four momenta of the initial and final fermions, respectively. m and m' denote the mass of the fermion for the vacuum polarization and the mass of the external fermion. a , b , c and q can be written in terms of k and p as

$$q = p' - p, \quad k = a - b, \quad k' = a - c, \quad q = k - k'.$$

Therefore, the S-matrix can be written as

$$S = ie^2 mm' g^2 \epsilon_\mu^\lambda(k) \epsilon_\nu^{\lambda'}(k') \frac{1}{q^2} \bar{u}(p') u(p) \int \frac{d^4 a}{(2\pi)^4} \frac{1}{a^2 - m^2} \frac{1}{(a - k)^2 - m^2} \frac{1}{(a - k')^2 - m^2} \times \text{Tr} [\gamma_\mu (\not{a} + m) \gamma_\nu ((\not{a} - \not{k} + m)((\not{a} - \not{k}' + m))] . \quad (5.8)$$

Since the term proportional to q does not contribute to the interaction, we can safely approximate in the evaluation of the trace and the a integration as

$$k' \approx k.$$

Now, we define the trace part as

$$N_{\mu\nu} = \text{Tr} [\gamma_\mu(\not{a} + m)\gamma_\nu((\not{a} - \not{k} + m)((\not{a} - \not{k}' + m))] \quad (5.9)$$

which can be evaluated as

$$N_{\mu\nu} = 4m[(k^2 - a^2 + m^2)g_{\mu\nu} + 4a_\mu a_\nu - 2a_\mu k_\nu - 2a_\nu k_\mu]. \quad (5.10)$$

Defining the integral by

$$I_{\mu\nu} \equiv \int \frac{d^4 a}{(2\pi)^4} \frac{N_{\mu\nu}}{(a^2 - m^2)[(a - k)^2 - m^2][(a - k')^2 - m^2]} \quad (5.11)$$

we can rewrite it using Feynman integral

$$I_{\mu\nu} = 2 \int \frac{d^4 a}{(2\pi)^4} \int_0^1 z dz \frac{N_{\mu\nu}}{[(a - kz)^2 - m^2 + z(1 - z)k^2]^3}. \quad (5.12)$$

Therefore, introducing the variable $w = a - kz$ we obtain the S-matrix as

$$S = 8ie^2 m^2 m' g^2 \epsilon_\mu^\lambda(k) \epsilon_\nu^{\lambda'}(k') \frac{1}{q^2} \bar{u}(p') u(p) \times \int \frac{d^4 w}{(2\pi)^4} \left[\frac{(-w^2 g_{\mu\nu} + 4w_\mu w_\nu)}{[w^2 - m^2 + z(1 - z)k^2]^3} + \frac{\{m^2 + k^2(1 - z^2)\} g_{\mu\nu} + 4k_\mu k_\nu z(1 - z)}{[w^2 - m^2 + z(1 - z)k^2]^3} \right]. \quad (5.13)$$

The first part of the integration can be carried out in a straightforward way using the dimensional regularization, and we find

$$\int \frac{d^4 w}{(2\pi)^4} \frac{(-w^2 g_{\mu\nu} + 4w_\mu w_\nu)}{[w^2 - m^2 + z(1 - z)k^2]^3} = \frac{i\pi^2 \Gamma(0) g_{\mu\nu}}{2\Gamma(3)} (4 - 4) = 0.$$

Thus, the two divergent parts just cancel with each other, and the cancellation here is not due to the regularization as employed in the vacuum polarization in QED, but it is a kinematical and thus rigorous result. The finite part can be easily evaluated [8], and therefore we obtain the S-matrix as

$$S = \frac{e^2}{8\pi} m^2 m' g^2 (\epsilon^\lambda \epsilon^{\lambda'}) \frac{1}{q^2} \bar{u}(p') u(p) \quad (5.14)$$

where we made use of the relation $k^2 = 0$ for free photon at the end of the calculation.

6 Renormalization Scheme for Gravity

At the present stage, it is difficult to judge whether we should quantize the gravitational field or not. At least, there is no experiment which shows any necessity of the quantization of the gravity. Nevertheless, it should be worth checking whether the gravitational interaction with fermions can be renormalizable or not. We know that the interaction of the gravity with fermions is extremely small, but we need to examine whether the graviton can stay massless or not within the perturbation scheme.

Here, we present a renormalization scheme for the scalar field theory which couples to fermion fields. The renormalization scheme for scalar fields is formulated just in the same way as the QED scheme since QED is most successful.

6.1 Vacuum Polarization of Gravity

First, we write the vacuum polarization for QED with the dimensional regularization, and the divergent contributions to the self-energy of photon can be described in terms of the vacuum polarization $\Pi_{QED}^{\mu\nu}(k)$ as

$$\begin{aligned}\Pi_{QED}^{\mu\nu}(k) &= i\lambda^{4-D}e^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\gamma^\mu \frac{1}{\not{p} - m} \gamma^\nu \frac{1}{\not{p} - \not{k} - m} \right] \\ &= \frac{e^2}{6\pi^2\epsilon} (k^\mu k^\nu - g^{\mu\nu} k^2) + \text{finite terms}\end{aligned}\tag{6.1}$$

where D is taken to be $D = 4 - \epsilon$. It is interesting to note that the apparent quadratic divergence disappears due to the gauge invariant dimensional regularization when evaluating the momentum integrations. This is important since, if there were any quadratic divergence terms present, then it would have caused serious troubles for the mass terms which cannot keep the gauge invariance in QED. The fact that the quadratic divergence terms can be erased by the proper dimensional regularization in QED is indeed related to the success of QED renormalization scheme.

On the other hand, the vacuum polarization for the gravity becomes

$$\begin{aligned}\Pi(k) &= i\lambda^{4-D}m^2g^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\frac{1}{\not{p} - m} \frac{1}{\not{p} - \not{k} - m} \right] \\ &= \frac{m^2g^2}{12\pi^2} \left\{ 3\Gamma(-1 + \frac{\epsilon}{2}) \left(m^2 - \frac{1}{6}k^2 \right) + \Gamma(\epsilon)k^2 \right\}\end{aligned}\tag{6.2}$$

This can be rewritten as

$$\Pi(k) = -\frac{1}{2}C_1k^2 - \frac{1}{2}C_2m^2\tag{6.3}$$

where

$$C_1 = -\frac{m^2g^2}{2\pi^2} \left(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{6} \right)\tag{6.4a}$$

$$C_2 = \frac{m^2 g^2}{2\pi^2} \left(\frac{2}{\epsilon} - \gamma + 1 \right). \quad (6.4b)$$

As can be seen, the second term in eq.(6.3) should correspond to the quadratic divergence term, and this time it cannot be erased by the dimensional regularization. However, this term can be safely eliminated by the counter term. Therefore, we add the following Lagrangian density as the mass counter-terms to the original Lagrangian density

$$\delta\mathcal{L} = \frac{1}{2}C_1\partial_\mu\mathcal{G}\partial^\mu\mathcal{G} - \frac{1}{2}\delta M\mathcal{G}^2 \quad (6.5)$$

where the constant δM is defined as

$$\delta M \equiv C_2 m^2. \quad (6.6)$$

Therefore, the total Lagrangian density of the gravity \mathcal{L}_G becomes

$$\mathcal{L}_G = \frac{1}{2}(1 + C_1)\partial_\mu\mathcal{G}\partial^\mu\mathcal{G} - \frac{1}{2}\delta M\mathcal{G}^2 = \frac{1}{2}\partial_\mu\mathcal{G}_r\partial^\mu\mathcal{G}_r - \frac{1}{2}\delta M\mathcal{G}_r^2 \quad (6.7)$$

where \mathcal{G}_r is the renormalized gravity field. This shows that the mass counter term cannot be renormalized into the wave function \mathcal{G} . However, the mass counter term in eq.(6.7) has a proper symmetry property of the gravity Lagrangian density, in contrast to the QED case where the mass term violates the gauge invariance. Therefore, the introduction of the mass counter term in the scalar field theory does not break the renormalization scheme of the present formulation.

6.2 Fermion Self Energy from Gravity

The fermion self energy term in QED is calculated to be

$$\Sigma_{QED}(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{1}{\not{p} - \not{k} - m} \gamma^\mu \frac{1}{k^2} = \frac{e^2}{8\pi^2\epsilon} (-\not{p} + 4m) + \text{finite terms}. \quad (6.8)$$

In the same way, we can calculate the fermion self-energy due to the gravity

$$\Sigma_G(p) = -im^2 g^2 \lambda^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\not{p} - \not{k} - m} \frac{1}{k^2} = \frac{m^2 g^2}{8\pi^2\epsilon} (-\not{p} + 4m) + \text{finite terms} \quad (6.9)$$

which is just the same as the QED case, apart from the factor in front. Therefore, the renormalization procedure can be carried out just in the same way as the QED case since the total fermion self energy term within the present model becomes

$$\Sigma(p) = \frac{1}{8\pi^2\epsilon} (e^2 + m^2 g^2) (-\not{p} + 4m) + \text{finite terms}. \quad (6.10)$$

6.3 Vertex Correction from Gravity

Concerning the vertex corrections which arise from the gravitational interaction and electromagnetic interaction with fermions, it may well be that the vertex corrections do not become physically very important. It is obviously too small to measure any effects of the higher order terms from the gravity and electromagnetic interactions. However, we should examine the renormalizability of the vertex corrections and can show that they are indeed well renormalized into the coupling constant. The vertex corrections from the electromagnetic interaction and the gravity can be evaluated as

$$\Lambda_{QED}(k, q) = i\lambda^{4-D} m g e^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\gamma_\mu \frac{1}{(\not{k} - \not{p} - m)(\not{k} - \not{p} - \not{q} - m)p^2} \gamma^\mu \right] \quad (6.11a)$$

$$\Lambda_G(k, q) = i\lambda^{4-D} m^3 g^3 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\frac{1}{(\not{k} - \not{p} - m)(\not{k} - \not{p} - \not{q} - m)p^2} \right]. \quad (6.11b)$$

We can easily calculate the integrations and obtain the total vertex corrections for the zero momentum case of $q = 0$ as

$$\Lambda(k, 0) = \Lambda_{QED}(k, 0) + \Lambda_G(k, 0) = \frac{mg}{\pi^2 \epsilon} (e^2 + m^2 g^2) + \text{finite terms} \quad (6.12)$$

which is logarithmic divergence and is indeed renormalizable just in the same way as the QED case.

6.4 Renormalization Procedure

Since the infinite contributions to the fermion self-energy and to the vertex corrections in the second order diagrams are just the same as the QED case, one can carry out the renormalization procedure just in the same way as the QED case. There is only one difference between QED and the gravity cases, that is, the treatment of the quadratic divergence in the vacuum polarization. In the QED case, the quadratic divergence terms should be eliminated by the dimensional regularization since the mass term violates the gauge invariance and thus one cannot consider the mass counter term in the QED Lagrangian density. On the other hand, in the gravity case, the quadratic divergence terms in the vacuum polarization can be canceled out by a mass counter term since the gravity is not the gauge field theory, and thus, there is no problem to introduce the mass counter term in the Lagrangian density. Further, the graviton is never bound and always in the free state, and therefore, the mass counter term in the gravity cancels the quadratic divergence contribution in a rigorous way. In this way, we can achieve a successful renormalization scheme for the gravity, even though we do not know any occasions in which the higher order contributions may become physically important.

7 Gravitational Interaction of Photon with Matter

From eq.(5.14), one finds that the gravitational potential $V(r)$ for photon with matter field can be written as

$$V(r) = -\frac{G_0 \alpha m_t^2 M}{2} \frac{1}{r} \quad (7.1)$$

where m_t and M denote the sum of all the fermion masses and the mass of matter field, respectively. α denotes the fine structure constant $\alpha = \frac{1}{137}$. In this case, the equation of motion for photon \mathbf{A}_λ under the gravitational field becomes

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{G_0 \alpha m_t^2 M}{2} \frac{1}{r} \right) \mathbf{A}_\lambda = 0. \quad (7.2)$$

Assuming the time dependence of the photon field \mathbf{A}_λ as

$$\mathbf{A}_\lambda = \boldsymbol{\epsilon}_\lambda e^{-i\omega t} A_0(\mathbf{r}) \quad (7.3)$$

we obtain

$$\left(-\nabla^2 - \frac{G_0 \alpha m_t^2 M}{2} \frac{1}{r} \right) A_0(\mathbf{r}) = \omega^2 A_0(\mathbf{r}). \quad (7.4)$$

This equation shows that there is no bound state for photon even for the strong coupling limit of $G_0 \rightarrow \infty$.

8 Conclusions

We have presented a new scheme of treating the gravitational interactions between fermions in terms of the Lagrangian density. The gravitational interaction appears always as the mass term and induces always the attractive force between fermions. In addition, there is an interaction between photon and the gravity as the fourth order Feynman diagrams. The behavior of photon under the gravitational field may have some similarity with the result of the general relativity, but the solution of eq.(7.4) is still to be studied in detail.

Also, we have presented a renormalization procedure which is essentially the same as the QED renormalization scheme. There is one important difference between the QED and the gravity cases, that is, the treatment of the quadratic divergence in the vacuum polarization. In QED, one has to eliminate the quadratic divergence terms by the regularization so as to keep the gauge invariance of the Lagrangian density. On the other hand, in the gravity case, the quadratic divergence terms can be canceled out by the mass counter term since it does not contradict with any important symmetry of the Lagrangian density. Therefore, the renormalization scheme of the gravity interaction is well justified, and thus the propagator of the gravity stays massless. Clearly, this is the most important point in the whole renormalization procedure.

In this paper, we have not decided whether the gravitational field should be quantized or not since there is no definite requirement from experiment for the quantization. At the

present stage, both of the evaluation of the gravitational interactions with fermions should be equally reasonable. However, for the quantized theory of gravitational field, one may ask as to whether there is any method to observe a graviton or not. The graviton should be created through the fermion pair annihilation. Since this graviton can propagate as a free graviton like a photon, one may certainly have some chance to observe it through the creation of the fermion pair. But this probability must be extremely small since the coupling constant is very small, and there is no enhancement in this process unless a strong gravitational field like a neutron star may rapidly change as a function of time.

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